Poisson factor matchings and allocations of optimal tail

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A *Poisson point process* (Ppp) of intensity 1 is a random point set ω in $\mathbb{R}^d$ such that for any measurable $A_i \subset \mathbb{R}^d$, the number $N(A_i)$ of points in A_i satisfies:

- $N(A_i)$ is independent for disjoint A_i;
- $$\mathbf{P}[N(A_i) = k] = \frac{C^k e^{-C}}{k!},\ C = \mathcal{L}(A_i).$$

Note that this has an isometry-invariant distribution, i.e. $N(A)$ has the same distribution as $N(g(A))$ for any isometry g) .

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We will need to condition on "$0 \in \omega$". Although this event has 0 probability, conditioning makes sense... This is the same as adding an extra point to the origin:

$\mathbb{P}[\omega \in A \,|\, 0 \in \omega] = \mathbb{P}[\omega \cup \{0\} \in A] =: \mathbb{P}_0[A]$

Perfectly matching two Ppp's

Given are two independent P.p.p's in $\mathbb{R}^d$. Match them so that everybody finds a pair. Pairs should find each other using the same, local rule.

More formally:
Find a function that almost surely defines a perfect matching and is

- *equivariant* (commutes with the isometries of $\mathbb{R}^d$),
- *measurable*.

The resulting matching is called a factor matching.

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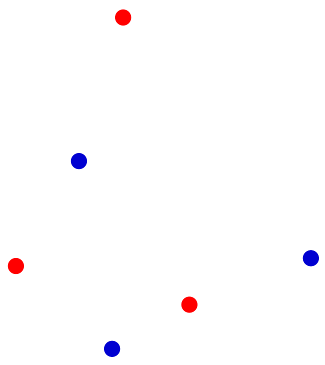
Practically, measurability means that the pair of a point $x \in \omega$ can be determined from a large neighborhood of x up to a small error, as a measurable function of the neighborhood.

An example: "Stable marriage"



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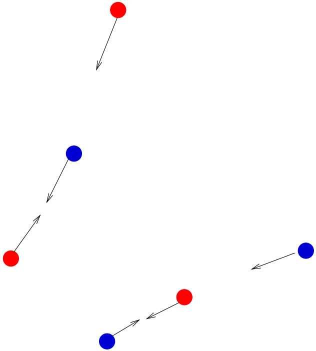

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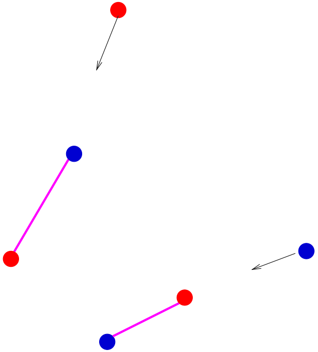

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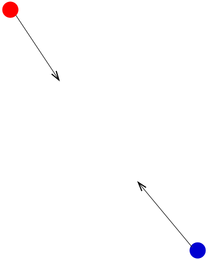

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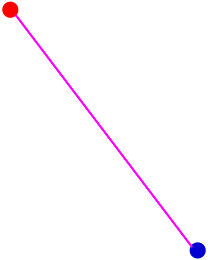

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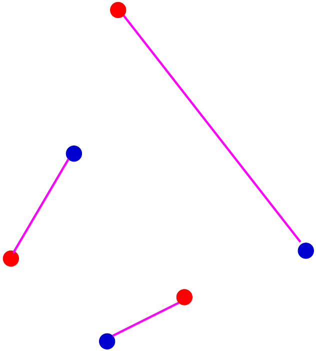

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If a blue and a red point are mutually closest to each other, match them.
Remove these pairs, and repeat the procedure for the remaining.

This is a factor matching. Holroyd and Peres have shown that in the limit every point gets matched this way.

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This is a *stable* matching scheme: *one cannot find a red and a blue point such that they are closer to each other than to their current pairs.*

It can be considered as an adaptation of the Gale and Shapley algorithm .

Our main question

What is the optimal tail one can obtain for the factor matching problem?

Question (Holroyd-Pemantle-Peres-Schramm, '07) Find a factor matching where $\mathbb{P}_0[X > r]$ decays as fast as possible. Here X is the distance of the origin from its pair.

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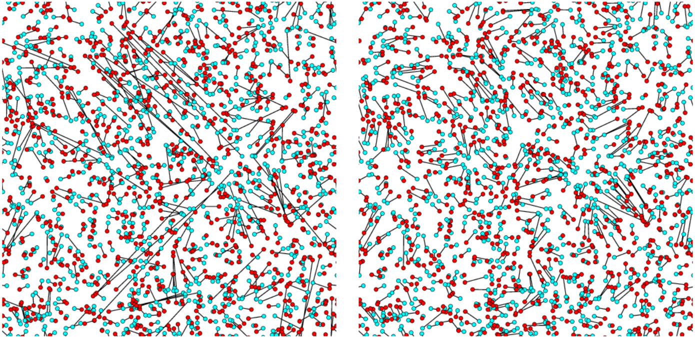
Stable matching is very far from optimal. (How sad...)



A trivial bound shows that the tail cannot be thinner than $c_0 \exp(-cr^d)$.

Theorem

For $d = 1, 2$, the fastest possible decay is $\mathbb{P}_0[X > r] < cr^{-d/2}$ up to the constant c, and there is a matching rule that has this tail. Here X is the distance of the origin from its pair.

Holroyd-Peres, '05; Meshalkin, '59; T., '08

From dimension 3, there is a drastic change in the behavior:

Theorem

(T., '08) Let $d \geq 3$. Then there is a matching factor such that $\mathbb{P}_0[X > r] < c_0 \exp(-cr^{d-2-\epsilon})$ for any $\epsilon > 0$.

From now on, we assume $d \geq 3$.

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Land division problem (allocation)

Farmers are distributed in the plane (or $\mathbb{R}^d$) according to P.p.p. Partition the world into parts of equal area, and assign these to the farmers. Use some "local" rule, and no central planning.

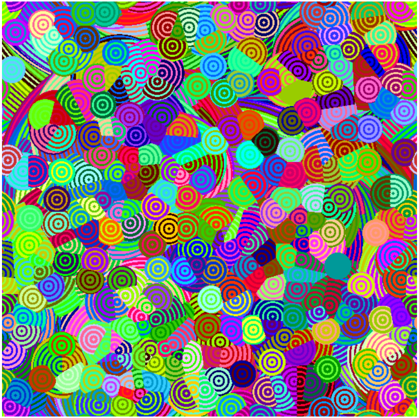
Math problem: Given P.p.p. with intensity 1, allocate to each point a unit area in an equivariant measurable way.

A rule for such an allocation will be called a (factor) allocation.

An application: an allocation gives rise to a shift-coupling between the point process and its Palm version. Thorisson

Stable allocation rule

One can define an allocation rule similarly to stable matching: Let the centers simultaneously start growing balls, so that at time t all balls have radius t. Let a center capture a point, if this is the first center to reach it, and the center has not captured a point set of volume 1 yet.



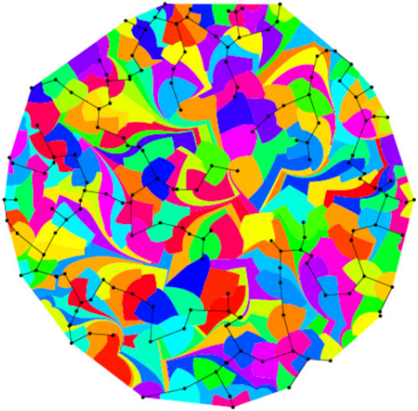
Find allocations of optimal tail

Similarly to matchings: find an optimal allocation rule, where the diameter of the cells decays as fast as possible.

Question(Holroyd-Peres, '05) What allocation rule does produce the fastest possible decay for $\mathbb{P}_0[\mathrm{diam}(\psi(0) \cup \{0\}) \geq R]$?

Just like for matchings, the stable allocation is far from optimal.

Various allocation rules got proposed...



Krikun, '07

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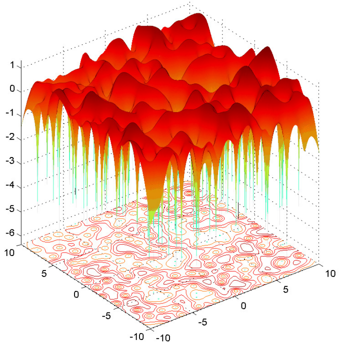

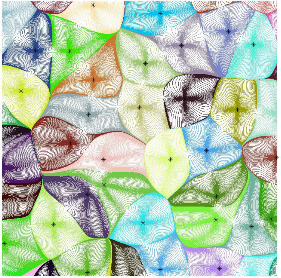

Gravitational allocation, Chatterjee, Peled, Peres, Romik, '10

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(Markó-T., '11) For $d \geq 3$ there exists an allocation rule for the P.p.p. in $\mathbb{R}^d$ that gives

$$\mathbb{P}_0[\mathrm{diam}(cell(0)) > r] \leq c_0 \exp(-cr^d).$$

So: there is an allocation that achieves the optimum, up to the constants.

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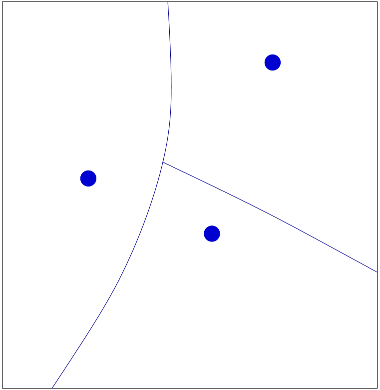
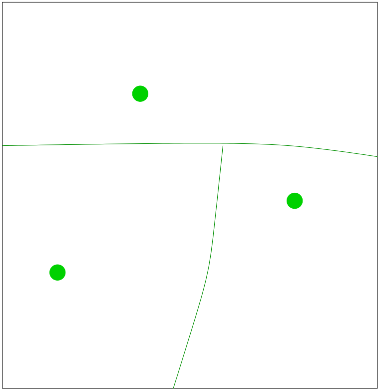
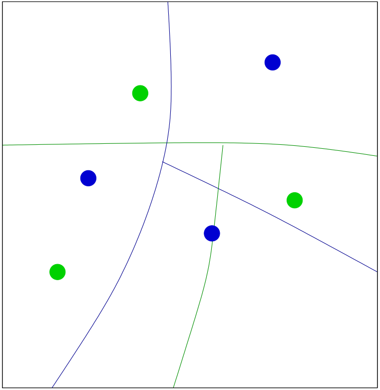
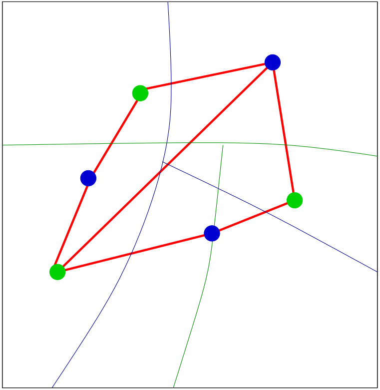
Is there any connection to the matching problem? Seemingly yes, but it was not clear, how...

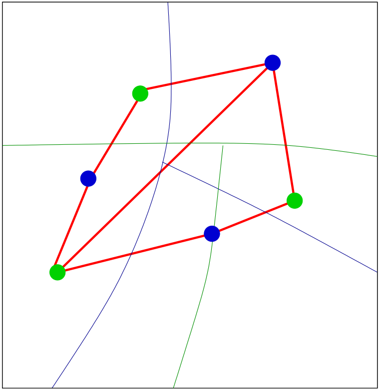
In the finite setting there is a direct connection, as observed by Ajtai, Komlós and Tusnády.











But in the setup of point processes, the graph that we would obtain is infinite, and we require the matching to be a factor! So we need some other version of Hall's criterion, if any.

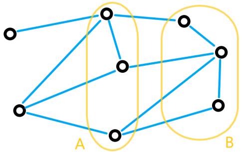
Definition: Given $(\Omega, \mathcal{B})$ Borel σ-algebra, a graph G on Ω is a *Borel graph* if its edge set is Borel in the product σ-algebra. Endow $(\Omega, \mathcal{B})$ with a probability measure μ. G is a *graphing* if

$$\int_A \deg_B(x) d\mu(x) = \int_B \deg_A(x) d\mu(x)$$

for every $A, B \in \mathcal{B}$.

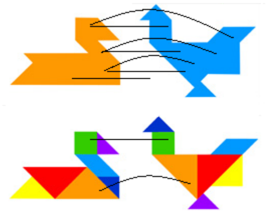
Example 1 A fixed finite graph with a uniform random vertex.

$$\sum_{x \in A} \deg_B(x) = \sum_{x \in B} \deg_A(x)$$



Example 2 Let $\Omega = S$ be a unit circle, μ uniform on S. Fix $\alpha \in [0, 2\pi]$. Let $x, y \in S$ be adjacent if rotation by $\pm\alpha$ takes x to y.

Example 3 "Tarski's circle squaring", equidecomposition questions



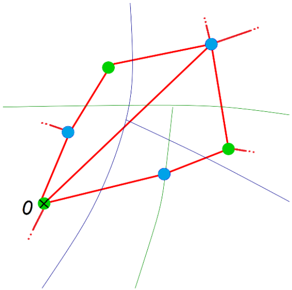
Note:
In Example 2, if α is irrational to π then there is no measurable perfect matching in this graphing (by ergodicity).
Even though Hall's criterion holds!

Example 4

ω_1 and ω_2 discrete point sets in $\mathbb{R}^d$, $0 \in \omega_1 \cup \omega_2$.
$A(x) :=$ the cell of $x \in \omega_i$ by some allocation rule.

$G(\omega_1, \omega_2)$ - a bipartite graph on $\omega_1 \cup \omega_2$, where x and y are adjacent if $A(x) \cap A(y) \neq \emptyset$.

$\Omega = \{G(\omega_1, \omega_2)\}$.



$G(\omega_1, \omega_2)$ and $G(\omega_1', \omega_2')$ are adjacent if

- there is an isometry ϕ of $\mathbb{R}^d$ with $\phi(\omega_1) = \omega_1'$, $\phi(\omega_2) = \omega_2'$,
- such that $0 (\in \omega_1 \cup \omega_2)$ and $\phi^{-1}(\omega_1')$ are adjacent in $G(\omega_1, \omega_2)$.

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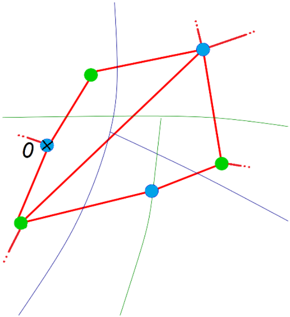

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A *fractional (perfect) matching* on a graph G is a function $\phi : E(G) \to [0,1]$ where $\sum_{w \sim v} \phi(\{v,w\}) = 1$ for every $v \in V(G)$. An equivalent of Hall's criterion is the existence of a fractional matching.

Theorem

(Bowen-Kun-Sabok, '21) Let $\mathcal{G}$ be a hyperfinite, one-ended bipartite graphing. Suppose that $\mathcal{G}$ has a measurable a.e. positive fractional perfect matching. Then it has a measurable perfect matching as well.

Definitions:

One-ended graphing – every component has one end, i.e., cannot be separated into ≥ 2 infinite components by a finite set.

Hyperfinite – for every $\varepsilon > 0$ there is an $A \subset \Omega$ of measure $< \varepsilon$ such that every component of $\Omega \setminus A$ is finite.

Theorem

(T., '21) For $d \geq 3$ there exists a factor perfect matching between two P.p.p. in $\mathbb{R}^d$ that gives

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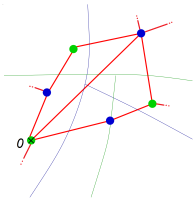
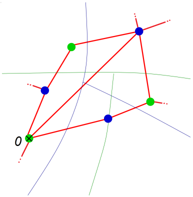
To prove it: Consider the graphing that we just defined with the Markó-T. allocation rule, and show that the assumptions of the Bowen-Kun-Sabok Theorem hold. The resulting perfect matching will have similar tail as the allocation.



Use the graphing defined before. Every degree is finite, by a property of the Markó-T. allocation rule used.

Define the weight of edge $\{\omega, \omega'\}$ as $\mathrm{Leb}(\mathrm{A}(0) \cap \mathrm{A}(\mathrm{v}))$, where $A(x) :=$ is the cell of a point.
This defines a measurable fractional perfect matching of the graphing.

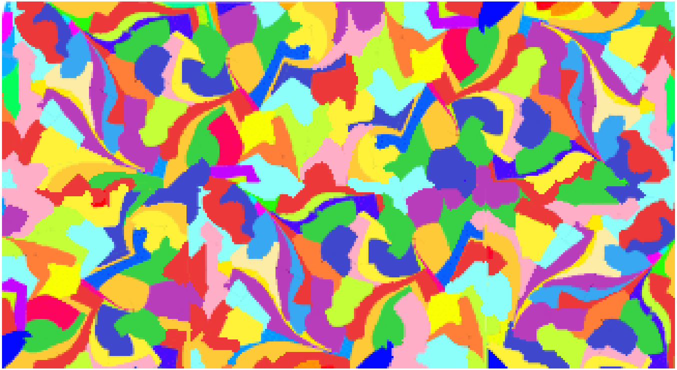


To prove one-endedness:
Note: the component of almost every point of the graphing is isomorphic to the graph in the respective configuration. So we need to prove that these graphs are one-ended.

Suppose not: then one could also find a finite set of blue points whose removal from the configuration graphs results in ≥ 2 infinite components.

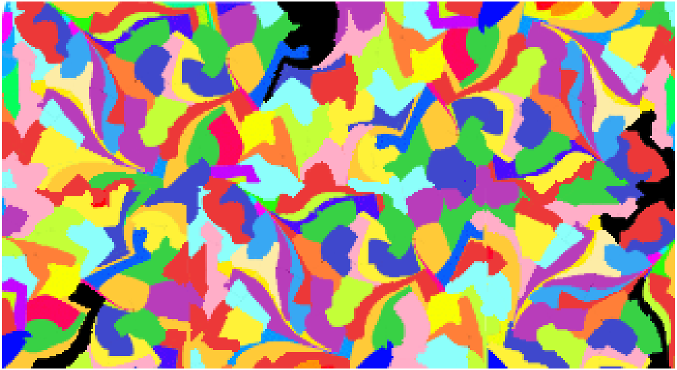
Then there would be a finite collection of (bounded) allocation cells whose removal from $\mathbb{R}^d$ results in more than one unbounded component. This is not possible.

Hyperfiniteness



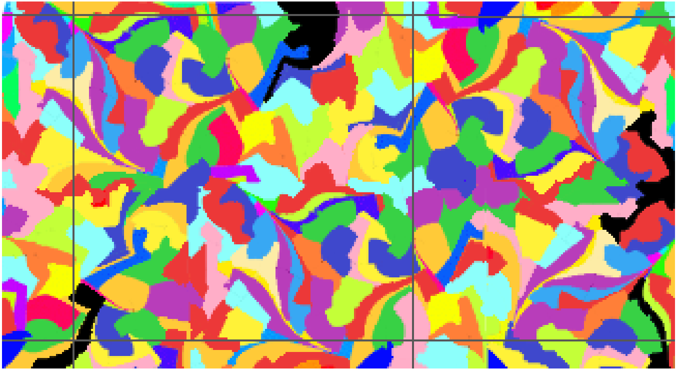
We are looking for a subset of density $< \varepsilon$ that splits the graph into finite pieces. We will search for it in the two color classes in parallel.

Hyperfiniteness



First remove cells that have diameter $> r$.

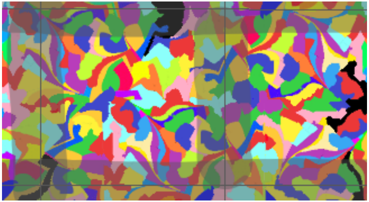
Hyperfiniteness



Then take a partition of $\mathbb{R}^d$ to convex *pieces*, each of which contains a ball of radius N ($N >> r$). Make this partition be a factor of the configuration.

Remove cells that intersect the boundary ∂P of some piece P.

Hyperfiniteness



The retained cells have

1. have diameter $\leq r$,
2. they have some point outside of r-neighborhood of $\cup \partial P$.

So all the retained cells are disjoint from $\cup \partial P$. Hence all the components in G induced by the retained cells (in the two classes of bipartition) are finite.

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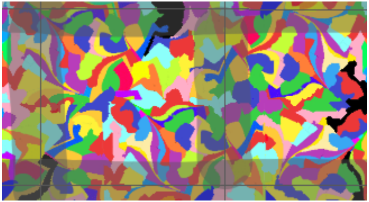

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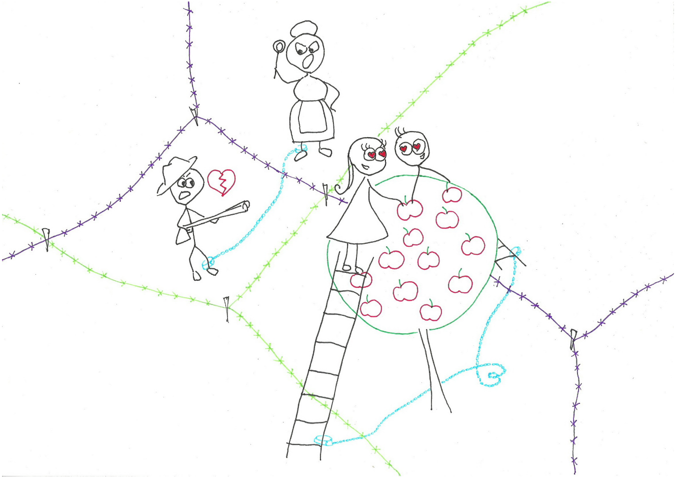
Both the "density" of (1) and (2) is arbitrarily close to 1 if r and N are large enough. $\diamond$

Thank you!



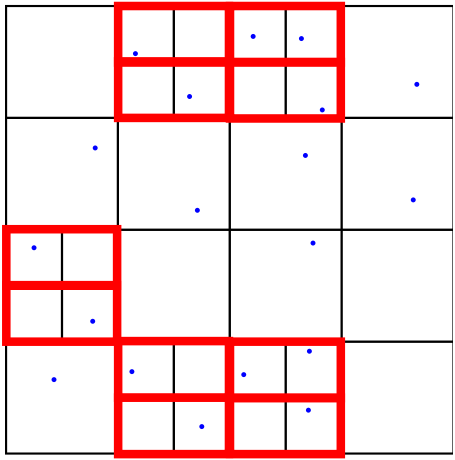
Optimal land division in a box

Let n points be uniformly independently distributed in a cube of volume n. Assign a volume 1 *cell* to each of them, partitioning the cube. Make the average diameter of cells minimal.

Optimal solution was given by Ajtai-Komlós-Tusnády , the "AKT algorithm".

The average diameter is $\log^{1/2} n$ for dimension 2, and finite for dimension ≥ 3.

The AKT algorithm



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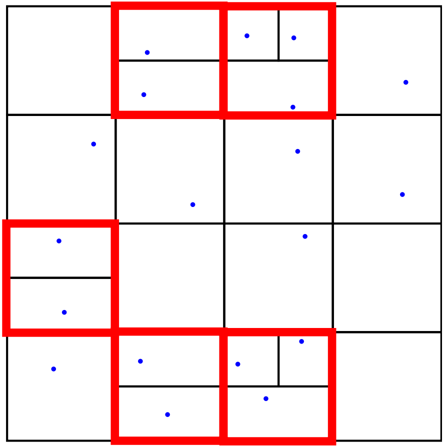

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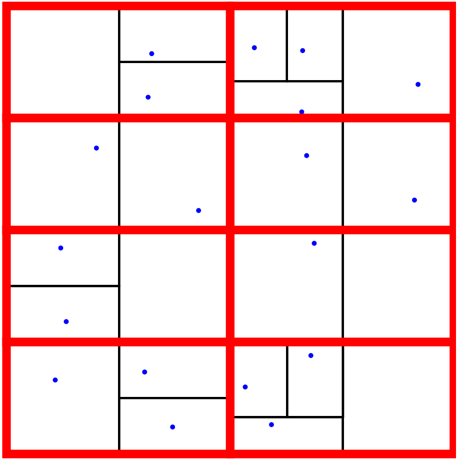

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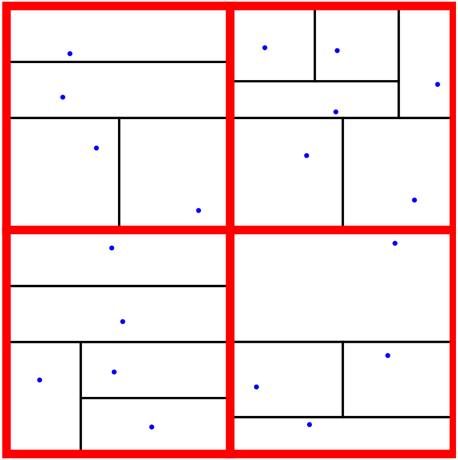

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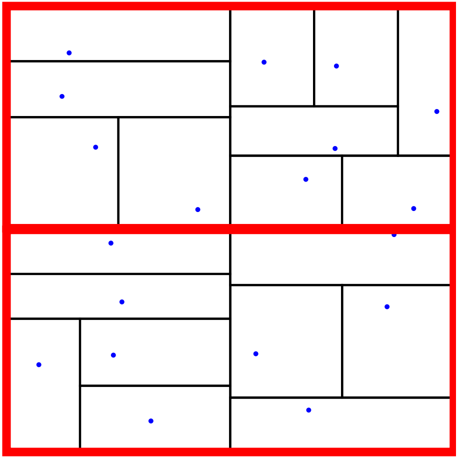

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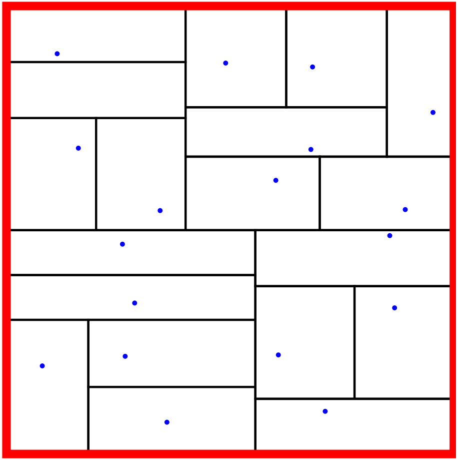

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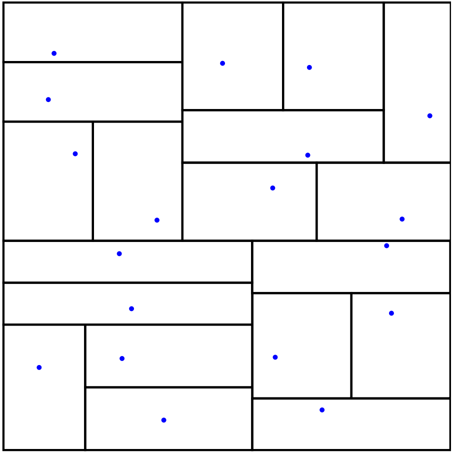

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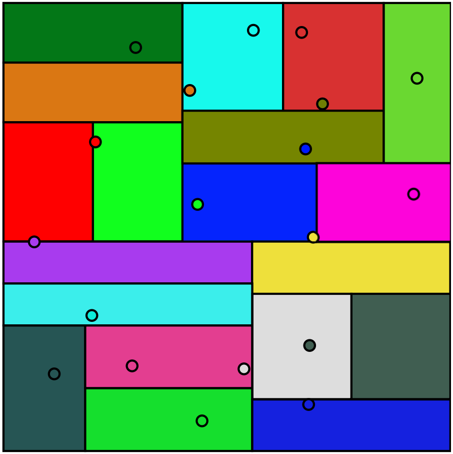

Sketch of proof for the optimal allocation

Theorem

(Markó-T., '11) For $d \geq 3$ there exists an allocation rule for the P.p.p. in $\mathbb{R}^d$ that gives

$$\mathbb{P}_0[\mathrm{diam}(cell(0)) > r] \leq c\exp(-cr^d).$$

Sketch of proof: Fix the point configuration ω.

Given $n \in \mathbb{Z}^+$, $v \in [0, 2^n)^d$, partition $\mathbb{R}^d$ to the cubes of $v + 2^n\mathbb{Z}^d$. For each of these cubes, allocate cells to the points of ω in the cube, using the AKT algorithm. For n large, most cell sizes are close to 1.

For $x \in \omega$, let $f_{n,v}(x, .)$ be the indicator fuction of the cell of x.

Sketch of proof for the optimal allocation

Consider $f_n(x,.) = \frac{1}{2^{dn}} \int_{[0,2^n)^d} f_{n,v}(x,.)dv$.

Claim: $\int_{\mathbb{R}^d} f_n$ is close to 1.

By an analysis of the AKT algorithm, $\mathrm{supp}(f_n)$ has good tail.

There exist an L^1-limit $f^{\omega,x}$ for the $f_n(x,.)$.

Claim: $\mathrm{supp}(f^{\omega,x})$ has good tail, $\int_{\mathbb{R}^d} f^{\omega,x}$ is 1, and $\sum_{x\in\omega} f^{\omega,x} = 1$ almost everywhere.

Sketch of proof for the optimal allocation

Consider $f_n(x,.) = \frac{1}{2^{dn}} \int_{[0,2^n)^d} f_{n,v}(x,.)dv$.

Claim: $\int_{\mathbb{R}^d} f_n$ is close to 1.

By an analysis of the AKT algorithm, $\mathrm{supp}(f_n)$ has good tail.

There exist an L^1-limit $f^{\omega,x}$ for the $f_n(x,.)$.

Claim: $\mathrm{supp}(f^{\omega,x})$ has good tail, $\int_{\mathbb{R}^d} f^{\omega,x}$ is 1, and $\sum_{x\in\omega} f^{\omega,x} = 1$ almost everywhere.

The family $\{f^{\omega,x} : x \in \omega\}$ was defined as a factor.

We obtain an allocation by suitably replacing the $f^{\omega,x}$ by indicator functions of sets within $\mathrm{supp}(f^{\omega,x})$. ◇